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On the modified Green operator integral for polygonal, polyhedral and other non-ellipsoidal inclusions

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Abstract

A "strange" particularity of polyhedral inclusions and of fibres of regular polygonal cross sections has been recently stressed in the literature. For respectively fully or transversally isotropic elasticity of the embedding material, they have a mean and a central Green operator integral both equal to the uniform one of respectively the sphere or the cylindrical fibre. In using the Radon transform (RT) method, this particularity is here shown to be shared by much larger shape types in the same limits of material elasticity symmetry. As a subcase, even more shape types fulfill the similar particularity for material linear properties of second-rank characteristic tensor, such as thermal conductivity, magnetic or dielectric properties. When calculated using the RT method, the modified Green operator integral at any interior point of a bounded domain (inclusion) takes the form of a weighted average over an angular distribution of a single elementary operator. The weight function is geometrically defined from the characteristic function of the domain, and the four-rank or second-rank elementary operator depends on the material linear property of concern. The RT method simply shows that the noticed particularity is due to matching symmetry between the inclusion shape (through the weight function) and the material property (through the elementary operator). The general geometrical characteristics of the inclusion shapes belonging to these sphere-class and cylindrical-fibre-class are specified, and some remarkable shapes of these classes are commented.

Keywords: Radon transform; Green operator integral; Inclusion shapes; Elasticity; Conductivity

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1. Introduction

Several quite recent papers (Mura et al., 1994, 1995; Rodin, 1996; Mura, 1997; Lubarda and Markenscoff, 1998; Nozaki and Taya, 2001; Kawashita and Nozaki, 2001), are concerned with remarkable characteristics of particular non-ellipsoidal inclusions, embedded in an infinite medium (the matrix), of which the considered property is isotropic elasticity. In these works, attention is mainly drawn on polygonal or polyhedral inclusions, either of convex or non-convex shapes. Two questions are the most often addressed. One is the possibility for some non-ellipsoidal inclusions to have a uniform interior Eshelby tensor, or modified Green operator integral, as ellipsoids have. Papers presenting inclusions fulfilling this "ellipsoid property" (Mura et al., 1994, 1995; Mura, 1997) are still controversially debated (Rodin, 1996; Lubarda and Markenscoff, 1998). More surely established is a weaker particularity, called "strange property" (Nozaki and Taya, 2001; Kawashita and Nozaki, 2001), noticed for both some fibres of regularly polygonal cross section and for some regular and semi regular polyhedral inclusions. These authors have pointed that, for such shapes, the Eshelby tensor (thus the modified Green operator integral) at the inclusion centre, and its mean value over the inclusion, are both equal to the (uniform) interior one of the fibre of circular cross section (cylindrical fibre) and of the sphere respectively. We here show that making use of the Radon transform method (Gel'fand et al., 1966), one simply establishes that, in the respective limits of transversally and fully isotropic elasticity, the above defined "strange property", denoted "$\widetilde{S}$—property" in the following, is shared by larger classes of inclusions than 2D-polygonal fibres or 3D-polyhedral inclusions. It is also shown, in the same symmetry limits, that for material linear properties defined by a second-rank tensor, such as dielectric or magnetic properties, as well as thermal conductivity, the $\widetilde{S}$—property holds for larger shape classes than in elasticity. As recently presented in the elasticity context (Franciosi and Lormand, 2004), using the Radon transform method, the modified Green operator integral at any interior point of an inclusion takes the form of a weighted average, over an angular distribution, of a single elementary operator. The weight function is geometrically defined from the characteristic function of the domain, and the four-rank or second-rank elementary operator depends on the material property of concern. The method allows to specify that the requested characteristic for the concerned shapes to fulfill the $\widetilde{S}$—property is to have a central and a mean weight function which exhibit a same fundamental symmetry type, what defines a sphere class and a cylindrical fibre class of shapes, with respect to the $\widetilde{S}$—property, that we here call "$\widetilde{S}$—class ". The $\widetilde{S}$—property is shown due to matching symmetry between the inclusion shape and the material property. Among these shapes are consequently all those possessing the same symmetry as a regular polygon or polyhedron which fulfils the $\widetilde{S}$—property. Section 2 recalls the Radon transform method, for four-rank **C** elasticity tensors at first, second-rank tensors showing up as a sub-case. Section 3 presents the geometric characteristics of the (2D and 3D) $\widetilde{S}$—class shapes. Section 4 comments on some remarkable shapes of the $\widetilde{S}$—classes, and Section 5 concludes.

2. The Green operator integral from the Radon transform method

We here extend the elastic framework considered in Franciosi and Lormand (2004), to any linear properties of an infinite homogeneous material either obeying a **C** four-rank tensor, such as for elasticity moduli, or a second-rank one, such as for dielectric, magnetic or thermal conductivity properties. The related modified $\mathbf{\Gamma}(\mathbf{r}-\mathbf{r}')$ Green tensor, of same rank as **C**, is classically defined from the second derivatives of a $\mathbf{G}(\mathbf{r}-\mathbf{r}')$ Green function as

$$\mathbf{C}:\mathbf{\Gamma}(\mathbf{r}-\mathbf{r}')=\mathbf{C}:(-\partial\partial\mathbf{G}(\mathbf{r}-\mathbf{r}'))=-\delta(\mathbf{r}-\mathbf{r}')\Delta. \tag{1}$$

In Eq. (1),"$\partial\partial$" stands for "$\partial^2/(\partial x_p\partial x_q)$", acting on $\mathbf{r}=(x_1,x_2,x_3)$, Δ is the Kronecker tensor or unity whether **C** is a four-rank or a second-rank tensor, and $\boldsymbol{\delta}(\mathbf{r})$ is the delta (generalised) function in R^3. We denote $\mathbf{t}^V(\mathbf{r})=\int_V\mathbf{\Gamma}(\mathbf{r}-\mathbf{r}')\mathrm{d}r'$ the modified Green operator integral over V, for **r** either inside or outside V.

Here, only $\mathbf{r}$ interior (including boundary) points of V will be considered. The volume average of $\mathbf{t}^V(\mathbf{r})$ over V reads $\overline{\mathbf{t}^V} = \frac{1}{v}\int_V \mathbf{t}^V(\mathbf{r})\mathrm{d}\mathbf{r}$. Unless differently specified, V here is a general regular [1], convex or not, simply connected, bounded domain, including polyhedrons and fibres of polygonal cross sections as limit cases. We then make use of the expressions of $\mathbf{t}^V(\mathbf{r})$ and $\overline{\mathbf{t}^V}$ in terms of their inverse (3D) Radon transform, which write, with $\boldsymbol{\omega} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$, $\mathrm{d}\omega = \sin(\theta)\mathrm{d}\theta\,\mathrm{d}\phi$

$$\mathbf{t}^V(\mathbf{r}) = \int_\Omega \mathbf{t}^P(\omega)\psi_V(\omega,\mathbf{r})\mathrm{d}\omega; \quad \overline{\mathbf{t}^V} = \int_\Omega \mathbf{t}^P(\omega)\overline{\psi_V}(\omega)\mathrm{d}\omega \tag{2}$$

In Eq. (2), $\mathbf{t}^V(\mathbf{r})$ and $\overline{\mathbf{t}^V}$ are angular averages of $\mathbf{t}^P(\omega)$ elementary operators, of respectively $\psi_V(\omega,\mathbf{r})$ and $\overline{\psi_V}(\omega)$ weight functions, with $\overline{\psi_V}(\omega) = \frac{1}{v}\int_V \psi_V(\omega,\mathbf{r})\mathrm{d}\mathbf{r}$, and [2]

$$\text{(i)}\;\; \psi_V(\omega,\mathbf{r}) = \frac{1}{8\pi^3}\int_V \left(\int_{k=0}^{\infty} k^2 \exp^{-ik\boldsymbol{\omega}(\mathbf{r}-\mathbf{r}')}\mathrm{d}k\right)\mathrm{d}\mathbf{r}'; \quad k = |\mathbf{K}|, \tag{3a}$$

$$\text{(ii)}\;\; \begin{cases} t^P_{pqjn}(\omega) = \left((M^{-1})_{pj}(\omega)\omega_q\omega_n\right)_{(pq),(jn)}; \quad M_{mp}(\boldsymbol{\omega}) = C_{mipj}\omega_j\omega_i \\ \qquad\text{or} \\ t^P_{qn}(\omega) = ((M^{-1})(\omega)\omega_q\omega_n); \quad M(\boldsymbol{\omega}) = C_{ij}\omega_j\omega_i. \end{cases} \tag{3b}$$

2.1. The morphological features of V and of the $\psi_V(\omega,\mathbf{r})$ weight function

Next introduce a (x,y,z) frame with $Oz//\mathbf{K}//\boldsymbol{\omega}$, such that $\mathbf{K}\cdot\mathbf{r} = k\boldsymbol{\omega}\cdot\mathbf{r} = kz$, and $s_V(z,\omega)$ is the planar section area of V through $\mathbf{r}$ and of $\boldsymbol{\omega}$-normal, by the plane of $z = \boldsymbol{\omega}\cdot\mathbf{r}$ equation. With $s''_V(z,\omega) = \frac{\partial^2}{\partial z^2}s_V(z,\omega)$, the weight function in Eq. (3a) takes the form

$$\psi_V(\omega,\mathbf{r}) = -\frac{1}{8\pi^3}\left(-\pi\int_{z'=D_V^-(\omega)}^{z'=D_V^+(\omega)}\left(\int_{s_V(z',\omega)}\mathrm{d}s_V(z',\omega)\right)\delta''(z-z',\omega)\mathrm{d}z'\right) = -\frac{1}{8\pi^2}s''_V(z,\omega), \tag{4a}$$

and its mean value over V reads, setting $d\mathbf{r} = \mathrm{d}s_V(z,\omega)\mathrm{d}z$, as for $\mathrm{d}\mathbf{r}'$ in Eq. (3a)

$$\overline{\psi_V}(\omega) = -\frac{1}{8\pi^2 v}\int_{D_V^-(\omega)}^{D_V^+(\omega)} s''_V(z,\omega)s_V(z,\omega)\mathrm{d}z = \frac{1}{8\pi^2 v}\int_{D_V^-(\omega)}^{D_V^+(\omega)}(s'_V(z,\omega))^2\mathrm{d}z. \tag{4b}$$

As shown in Fig. 1, $[D_V^-(\omega), D_V^+(\omega)] = 2D_V(\omega)$ is the breadth of V in the ω direction (i.e. the distance between the two opposite tangent planes to V, of $\boldsymbol{\omega}$-normal), which characterises the support function of V when convex ($D^V(\boldsymbol{\omega}) = \max_{\mathbf{r}\in V}(\boldsymbol{\omega}\cdot\mathbf{r})$), or of the convex hull of V otherwise. In R^2, the $D^V(\boldsymbol{\omega})$ breadth of V, either convex or not, equals its orthogonal projection, or brightness, $B_V(\omega^\perp)$ say, parallel to the $\boldsymbol{\omega}^\perp$ direction normal to $\boldsymbol{\omega}$, but not in R^3, since $D^V(\boldsymbol{\omega})$ and $B_V(\omega)$ functions are no more of same dimension. In R^2 (resp. R^3), for convex V domains, $B_V(\omega)$ equals the total $A_V(\omega)$ projection length (resp. area) of $\boldsymbol{\omega}$-normal, otherwise $A_V(\omega) > B_V(\omega)$. The $s_V(z,\omega)$ section area is the Radon transform of the $X_V(\mathbf{r})$ characteristic function of V, ($X_V(\mathbf{r}) = 1$ for $\mathbf{r}$ in V and 0 if not), and its inverse Radon transform (Gel'fand et al., 1966) yields

$$X_V(\mathbf{r}) = -\frac{1}{8\pi^2}\int_\Omega s''_V(z,\omega)\mathrm{d}\omega = \int_\Omega \psi_V(\omega,\mathbf{r})\mathrm{d}\omega = 1 \tag{5a}$$

[1] Non strictly zero product of the principal radii of curvature at any point on the V surface.
[2] '$(p,q),(j,n)$' in Eq. (3b) specifies the symmetry on the pairs of indices within brackets.

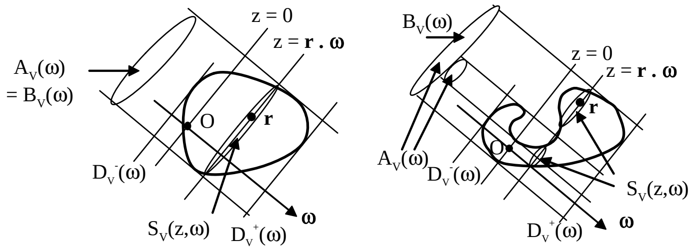

Fig. 1. A V (left convex, right non-convex) bounded domain, a $(0, \boldsymbol{\omega}, z(\boldsymbol{\omega}))$ coordinate system, the $S_V(z,\omega)$ section through $\mathbf{r}$, the $[D_V^-(\omega),\ D_V^+(\omega)]$ breadth, the $B_V(\omega)$ brightness, the $A_V(\omega)$ total projected area, in $\boldsymbol{\omega}$ direction.

$$\frac{1}{v}\int_V X_V(\mathbf{r})\mathrm{d}\mathbf{r} = -\frac{1}{8\pi^2}\int_\Omega\left(\int_{D_V^-(\omega)}^{D_V^+(\omega)} s_V''(z,\omega)s_V(z,\omega)\mathrm{d}z\right)\mathrm{d}\omega = \int_\Omega \overline{\psi_V}(\omega)\mathrm{d}\omega = 1 \tag{5b}$$

For an ellipsoidal inclusion of $2D_{\rm ell}(\omega)$ breadths, $s_{\rm ell}(z,\omega) = \frac{3v}{4D_{\rm ell}(\omega)}\left(1-\left(\frac{z}{D_{\rm ell}(\omega)}\right)^2\right)$ and $s''_{\rm ell}(z,\omega) = -\frac{3v}{2D_{\rm ell}(\omega)^3}$, $\forall\,\mathbf{r}$. Thus, the weight function uniformly writes in V

$$\psi_{\rm ell}(\omega) = \left(-\frac{1}{8\pi^2}\right)\left(-\frac{3v}{2D_{\rm ell}(\omega)^3}\right) = \left(\frac{3}{4\pi}\right)^2\left(\frac{v}{3D_{\rm ell}(\omega)^3}\right) = \frac{R_{\rm ell^*}(\omega)^3}{3v^*} = \frac{1}{3v^*D_{\rm ell}(\omega)^3}, \tag{6}$$

since for any convex V domain in R^n, $\int_\Omega \frac{1}{3D_V(\omega)^3}\mathrm{d}\omega = \int_\Omega \frac{R^3_{V^*}(\omega)}{3}\mathrm{d}\omega = v^*$, with v^* the volume of the V^* reciprocal convex body of V and $R_{V^*}(\omega) = \frac{1}{D_V(\omega)}$ its radial function (Santalo, 1976). For a sphere of radius R, $s_{\rm sph}(z,\omega) = \pi(R^2 - z^2)$, $s''_{\rm sph}(z,\omega) = -2\pi$ and $\psi_{\rm sph}(\omega,\mathbf{r}) = \frac{1}{4\pi} = \psi_{\rm sph}(\omega)$, $\forall\,(\mathbf{r},\boldsymbol{\omega})$. From Eq. (4a), $\psi_V(\omega,\mathbf{r})$ is even on Ω, i.e. $\psi_V(\omega,\mathbf{r}) = \psi_V(-\omega,\mathbf{r})\ \forall\,\mathbf{r}$, for any V domain. With $-V$ the centrally reflected body of V (Nagel, 1993), one has $\psi_{-V}(\omega,-\mathbf{r}) = \psi_V(\omega,\mathbf{r})$. In particular $\psi_{-V}(\omega,\mathbf{0}) = \psi_V(\omega,\mathbf{0})$ and $\overline{\psi_{-V}}(\omega) = \overline{\psi_V}(\omega)$. For a fibre-like inclusion, in all $\boldsymbol{\omega}$ directions not orthogonal to the fibre axis, the $2D_V(\omega)$ breadths are infinite and the related $\psi_{\rm fib}(\omega,\mathbf{r})$ terms are zero. For a circular cross section, the cylindrical fibre identifies to a spheroid of infinite aspect ratio, therefore of uniform $\psi_{\rm cfib}(\theta,\phi,\mathbf{r}) = \psi_{\rm cfib}(\theta,\phi)$ interior weight function, equal to $\frac{1}{2\pi}$ around the fibre axis, and zero otherwise. These results in R^3 for fibres are paradoxically more simply obtained than considering a 2D problem in R^2, the Radon inversion formula in even spaces (see Appendix A) being more complicated than in odd ones (Gel'fand et al., 1966; Natterer, 1986).

2.2. Characteristics of the $\mathbf{t}^P(\boldsymbol{\omega})$ elementary operator of material properties

As recalled in Franciosi and Lormand (2004) for elasticity, the $\mathbf{t}^P(\omega)$ elementary operator identifies to the uniform modified Green operator integral of a platelet V domain of same $\boldsymbol{\omega}$ small axis orientation. This also holds for second rank $\mathbf{C}$ tensors.

Elasticity (four-rank $\mathbf{C}$ *tensor).* Considering the $(0,0)$ oriented $\mathbf{t}^P(0,0)$ elementary operator for $(\theta,\phi) = (0,0)$, one has, for $\mathbf{M}$ in Eq. (3b), $M_{mp} = C_{m3p3}$, for $\mathbf{C}$ expressed in the operator axes frame, $\mathbf{C}_{(0,0)}$ say. Thus the $\mathbf{M}$ 3×3 symmetric submatrix of $\mathbf{C}_{(0,0)}$ and the corresponding $t^P_{p3j3}(0,0)$ non-zero terms (which also make a $\Delta\mathbf{t}$ symmetric 3×3 matrix) write, in terms of $\mathbf{C}$ and $\mathbf{N} = \mathbf{M}^{-1}$

$$\mathbf{M} = \begin{vmatrix} C_{1313} & C_{1323} & C_{1333} \\ C_{1323} & C_{2323} & C_{2333} \\ C_{1333} & C_{2333} & C_{3333} \end{vmatrix}_{0,0}; \quad \Delta\mathbf{t} = \begin{vmatrix} t_{1313} & t_{1323} & t_{1333} \\ t_{1323} & t_{2323} & t_{2333} \\ t_{1333} & t_{2333} & t_{3333} \end{vmatrix} = \begin{vmatrix} \frac{N_{11}}{4} & \frac{N_{12}}{4} & \frac{N_{13}}{2} \\ \frac{N_{12}}{4} & \frac{N_{22}}{4} & \frac{N_{23}}{2} \\ \frac{N_{13}}{2} & \frac{N_{23}}{2} & N_{33} \end{vmatrix}. \tag{7}$$

From $\mathbf{C}_{(0,0)}$, the $\mathbf{C}_{(\theta,\phi)}$ moduli in the $\mathbf{t}^P(\theta,\phi)$ operator frame are obtained by the $R(\theta,\varphi)$ rotation

$$R(\theta,\phi)=R(\omega)=\begin{vmatrix} -\omega_1\omega_3/(1-\omega_3^2)^{1/2} & \omega_2/(1-\omega_3^2)^{1/2} & \omega_1 \\ -\omega_2\omega_3/(1-\omega_3^2)^{1/2} & -\omega_1/(1-\omega_3^2)^{1/2} & \omega_2 \\ (1-\omega_3^2)^{1/2} & 0 & \omega_3 \end{vmatrix}. \tag{8}$$

Conversely, the expression in the same, $\mathbf{C}_{(0,0)}$ say, frame of all (θ,ϕ)-oriented $\mathbf{t}^P(\omega)$ elementary operators, in order to calculate $\mathbf{t}^V(\mathbf{r})$, is obtained by the inverse $R^{-1}(\theta,\varphi)=R^t(\theta,\varphi)$ rotation. For elastic isotropy, the $\mathbf{C}$ frame identification is made useless since in all frames $C_{3333}=2\mu(1-\nu)/(1-2\nu)$ and $C_{1313}=C_{2323}=\mu$. The non-zero terms of $\Delta\mathbf{t}$ are $N_{11}=1/C_{1313}$, $N_{22}=1/C_{2323}$, $N_{33}=1/C_{3333}$, and those of $\mathbf{t}^P(\theta,\phi)$ in the (θ,ϕ) operator frame are

$$\mathbf{t}^P_{3333}=\frac{1-2\nu}{2\mu(1-\nu)}=\frac{3}{K+4\mu}=\frac{1}{\mu}-\frac{1}{2\mu(1-\nu)}=B+A,\quad \mathbf{t}^P_{((2,3),(2,3))}=\mathbf{t}^P_{((3,1),(3,1))}=\frac{1}{4\mu}=\frac{B}{4}, \tag{9}$$

for all (θ,ϕ) orientations. In a same $(0,0)$ frame, the $\mathbf{t}^P(\omega)$ $(=\mathbf{t}^P(-\omega))$ operators thus rearrange as

$$t^P_{pqjn}(\omega)=A\delta^A_{pqjn}(\omega)+B\delta^B_{pqjn}(\omega);\quad \delta^A_{pqjn}(\omega)=\omega_j\omega_p\omega_n\omega_q;\quad \delta^B_{pqjn}(\omega)=(\Delta_{jp}\omega_n\omega_q)\big|_{(p,q),(j,n)}. \tag{10a}$$

The four-rank elementary $\mathbf{t}^P(\theta,\phi)ij/kl$ symmetric operator terms which then remain non-zero after ϕ and $\cos\theta$ integrations [3] are given in Table 1.

These 9 terms are defined by even trigonometric functions of the form $f^{(p,q)}(\phi)t^{(r,s)}(\theta)=\cos^{2p}\phi\sin^{2q}\phi\cos^{2r}\theta\sin^{2s}\theta$, non-vanishing upon integration. Owing to the five $(0,1)$, $(1,0)$, $(1,1)$, $(0,2)$ and $(2,0)$ possible values taken by both the (p,q) and (r,s) factor pairs, all terms can be expressed in using only two of the five $f^{(p,q)}(\phi)$ functions, and two of the five $t^{(r,s)}(\theta)$ ones, such as $\cos^2(.)$ and $\cos^4(.)$ for example. Simple dependency relations between these terms are

$$\begin{cases} t^P_{1111}(\omega)+t^P_{2222}(\omega)+2t^P_{1212}(\omega)=\dfrac{2A+3B}{2} \\ \displaystyle\sum_{i=1}^{3}\left(t^P_{iiii}(\omega)+\sum_{j=1}^{3}t^P_{iijj}(\omega)\right)=A+B \\ \displaystyle\sum_{i=1}^{3}\left(t^P_{iiii}(\omega)+\sum_{j=1}^{3}t^P_{ijij}(\omega)\right)=\mathrm{tr}(\mathbf{t}^P(\omega))=A+2B. \end{cases} \tag{10b}$$

Second-rank $\mathbf{C}$ *tensor*. In these cases, whatever is the material property, $M(\omega)$ in Eq. (3b) reduces to the C_{33} value in the $\boldsymbol{\omega}$ direction. If $\mathbf{C}$ is isotropic, one has $C_{ij}=C\delta_{ij}$ for all $\boldsymbol{\omega}$ directions, with C an appropriate constant for the considered property and

$$t^P_{qn}(\omega)=D\delta^D_{qn}(\omega);\quad D=1/C;\quad \delta^D_{qn}(\omega)=\omega_n\omega_q. \tag{11a}$$

[3] Integrals of products of odd powers of trigonometric functions vanish.

Table 1
Non zero terms of the elementary operator, with "cθ" and "sθ" for "cos(θ)"and "sin$'(\theta)$" (resp. ϕ)

	11	22	33	23	31	12
11	$As^4\theta c^4\phi$ $Bs^2\theta c^2\phi$	$As^4\theta c^2\phi s^2\phi$ 0	$As^2\theta c^2\theta c^2\phi$ 0			
22	$As^4\theta c^2\phi s^2\phi$ 0	$As^4\theta s^4\phi$ $Bs^2\theta s^2\phi$	$As^2\theta c^2\theta s^2\phi$ 0			
33	$As^2\theta c^2\theta c^2\phi$ 0	$As^2\theta c^2\theta s^2\phi$ 0	$Ac^4\theta$ $Bc^2\theta$			
23				$As^2\theta c^2\theta s^2\phi$ $B(s^2\theta s^2\phi + c^2\theta)/4$		
31					$As^2\theta c^2\theta c^2\phi$ $B(s^2\theta c^2\phi + c^2\theta)/4$	
12						$As^4\theta c^2\phi s^2\phi$ $Bs^2\theta/4$

The terms of $t^P_{qn}(\omega)$ which remain non-zero after ϕ and $\cos\theta$ integrations are the $t^P_{qq}(\omega)$ diagonal ones, which are identical to the B-parts of the $t^P_{qqqq}(\omega)$ terms given in Table 1, in replacing $B = 1/\mu$ by the appropriate D constant. These 3 terms can be expressed from a single $f^{(p,q)}(\phi)$ and $h^{(r,s)}(\theta)$ function pair over the four defined by the (p,q) and (r,s) factor pairs which then only take the $(0,1)$ and $(1,0)$ values. The dependency relation between these terms is

$$\sum_{i=1}^{3} t^P_{ii}(\omega) = \mathrm{tr}(\mathbf{t}^P(\omega)) = D. \tag{11b}$$

Eqs. (10a) and (11a) hold, with appropriate (A,B) or D scalar terms, in plane cross sections of a fibre-like inclusion, whatever its shape is, for a fibre-oriented transversal isotropy of the material property $\mathbf{C}$.

3. Application to polygonal, polyhedral, and other piece-wise regular inclusions

This discussion is first carried on 2D fibres, for at least transversally isotropic elasticity of the material with regard to the fibre axis, and then for 3D ones, for fully isotropic material elasticity. We first consider the elasticity four-rank tensor, and second-rank tensors next, as a sub-case.

3.1. Fibre-like inclusions

For $z//x3$ fibres, $\theta = \pi/2$ implies $\omega 3 = 0$, and the only $\mathbf{t}^P(\theta,\phi)$ involved elementary operators in $\mathbf{t}^V(\mathbf{r})$ and in $\overline{\mathbf{t}^V}$ are those for which $\theta = \frac{\pi}{2}$, $\phi \in [0, 2\pi]$. These operators have, according to Eq. (10a), six non-zero symmetrized $t^P_{ijkl}\left(\frac{\pi}{2},\phi\right)$ components related, $\forall\ \phi$, by Eq. (10b) (Table 2). From Eq. (11a), the two non-zero components of the $t^P_{ij}\left(\frac{\pi}{2},\phi\right)$ elementary operator correspond to the B-parts of the $t^P_{iiii}\left(\frac{\pi}{2},\phi\right)$ terms in Table 2, with D instead of B, and are linked by Eq. (11b).

For the fibre of circular cross section (the "cylindrical fibre" to be denoted "cfib"), the non-zero components of the $\mathbf{t}^{\mathrm{cfib}}$ Green operator integral thus correspond to ϕ integrals over 2π of the five even trigonometric functions of the form $F^{(p,q)} = \frac{1}{2\pi}\int_{\phi=0}^{2\pi}\cos^{2p}\phi\sin^{2q}\phi\,\mathrm{d}\phi$, with $p,q = 0,1,2$, $p+q = 1,2$ for the rank-four tensor, say five functions, only two of which being independent, say

Table 2
Non zero terms of a $\mathbf{t}^p(\pi/2,\phi)$ platelet operator with "$c\phi$" and "$s\phi$" for "$\cos(\phi)$" and "$\sin(\phi)$"

	11	22	33	23	31	12
11	$Ac^4\phi$ $Bc^2\phi$	$Ac^2\phi s^2\phi$ 0	0			
22	$Ac^2\phi s^2\phi$ 0	$Ac^4\phi$ $Bs^2\phi$	0			
33	0	0	0			
23				$(Bs^2\phi)/4$		
31					$(Bc^2\phi)/4$	
12						$Ac^2\phi s^2\phi$ $B/4$

$$\text{(i)}\ \ F^{(1,0)} = \frac{1}{2\pi}\int_{\phi=0}^{2\pi}\cos^2\phi\,\mathrm{d}\phi = \frac{1}{2\pi}\int_{\phi=0}^{2\pi}\frac{1}{1+tg^2\phi}\,\mathrm{d}\phi = \frac{1}{2}, \tag{12a}$$

$$\text{(ii)}\ \ F^{(2,0)} = \frac{1}{2\pi}\int_{\phi=0}^{2\pi}\cos^4\phi\,\mathrm{d}\phi = \frac{1}{2\pi}\int_{\phi=0}^{2\pi}\frac{1}{(1+tg^2\phi)^2}\,\mathrm{d}\phi = \frac{3}{8}, \tag{12b}$$

and with $F^{(0,1)} = 1 - F^{(1,0)} = \frac{1}{2}$, $F^{(1,1)} = F^{(1,0)} - F^{(2,0)} = \frac{1}{8}$, $F^{(0,2)} = 1 - (F^{(2,0)} + 2F^{(1,1)}) = \frac{3}{8}$.

Thus, substituting the integral values of Eq. (12) to the corresponding ϕ functions in Table 2

$$t^{\mathrm{cfib}}_{1111} = t^{\mathrm{cfib}}_{2222} = \frac{3A+4B}{8};\quad t^{\mathrm{cfib}}_{(2,3),(2,3)} = t^{\mathrm{cfib}}_{(3,1),(3,1)} = \frac{B}{8};\quad t^{\mathrm{cfib}}_{11,22} = \frac{A}{8}\,t^{\mathrm{cfib}}_{(1,2),(1,2)} = \frac{A+2B}{8}. \tag{13}$$

Each of the $F^{(p,q)}$ integrals is equal to a finite series value of the form

$$\frac{1}{N}\sum_{k=1}^{N}\cos^{2p}\left(\frac{2k\pi}{N}+\phi\right)\sin^{2q}\left(\frac{2k\pi}{N}+\phi\right) = \begin{cases} 1/2 \text{ for } (p,q) = (0,1),(1,0) \\ 3/8 \text{ for } (p,q) = (0,2),(2,0), \quad \forall\phi, \quad \forall N \neq 2,4. \\ 1/8 \text{ for } (p,q) = (1,1) \end{cases} \tag{14}$$

It is therefore immediate that any set of such N-platelet operators oriented along a set of $(\phi + 2k\pi/N)$, $k = 1,\ldots,N$, directions around the z axis, with $N \neq 2,4$, [4] is enough for any ϕ to give the $\mathbf{t}^{\mathrm{cfib}}$ operator, provided equal $(1/N)$ weights. The $N = 2$ or 4 values which do not fulfil this property in elasticity are stressed in Kawashita and Nozaki (2001). However, for the second-rank $\mathbf{C}$ tensors, Eq. (12a) being involved alone, only the first sum of Eq. (14) needs be fulfilled by a N-platelet operator set, what does not eliminate the $N = 4$ case. Now, one can immediately state that the $\widetilde{S}$—property also holds for continuous ϕ sets of elementary operators, provided that a $N \neq 2,4$ or $N \neq 2$, $\mathbf{C}$-rank dependent, fundamental $\frac{2\pi}{N}$ symmetry in ϕ is preserved. This means, the $\widetilde{S}$—property $\mathbf{t}(0) = \bar{\mathbf{t}} = \mathbf{t}^{\mathrm{cfib}}$ is expected to hold for any fibre-like inclusion of invariant cross section shape (thus weight function) by $\frac{2k\pi}{N}$ rotations around the fibre axis, not necessarily of polygonal cross section, as exemplified on Fig. 2 for various shapes of boundary elements replacing edges of regular N-polygons. In particular, elements of the illustrated (b) type, with $N = 3$, correspond to semi

[4] Since $\mathbf{t}^P(\omega) = \mathbf{t}^P(-\omega)$, odd $N = m$ values and corresponding $N = 2m$ values correspond to a same set of m elementary operators, counted twice for $N = 2m$. Thus $N = 1$ and 2 correspond to a single operator, $N = 3$ and 6 to 3 operators etc. In particular, $N = 4$ corresponds to a pair of orthogonal operators counted twice.

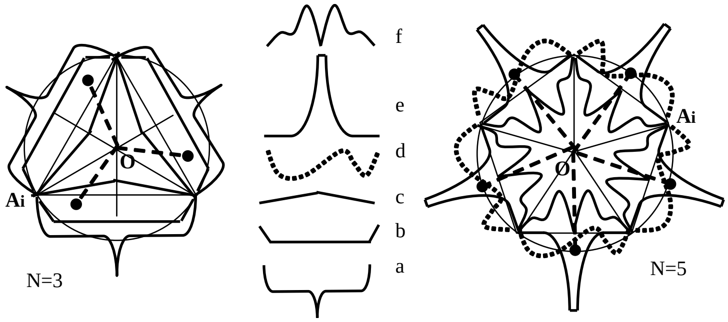

Fig. 2. Examples of sections shapes of the "cfib class". Left: for $N = 3$ using boundary element a, b or c on the trigonal fibre; Right: for $N = 5$ and using boundary elements d, e or f on the pentagonal fibre. Black dots represent N-sets of homologous points.

regular polygons, which are "regularly truncated" regular polygons, such as to preserve the fundamental N-symmetry. Also note that the not centrally symmetric elements of the (d) type yield shapes that have no axial symmetry. All these inclusions constitute a "cylindrical fibre class", with regard to the $\widetilde{S}$—property. This, so far assessed without any weight function calculations, formally amounts to write the modified Green operator integral at a $\mathbf{r}$ interior point of such a V shape

$$\mathbf{t}^V(\mathbf{r}) = \int_{\mathrm{O}} \mathbf{t}^P(\phi)\psi_V(\phi,\mathbf{r})\mathrm{d}\phi = \sum_{k=1}^{N} \int_{\Delta_{\mathbf{r}}^k \mathrm{O}} \mathbf{t}^P(\phi)\psi_{Vk}(\phi,\mathbf{r})\mathrm{d}\phi. \tag{15}$$

In Eq. (15), around $\mathbf{r}$, the O unit circle normal to the V fibre axis has been divided into $N\Delta_{\mathbf{r}}^k O$ arcs, $k = 1,\ldots,N$, intercepting the N regular boundary elements of the fibre cross section, between pairs of neighbouring vertices. For the central point of a cross section, $\Delta_{\mathbf{0}}^k \mathrm{O} = \Delta \mathrm{O}_{(N)} \forall\, k$, and the inclusion symmetry ensures the equality of all the $\psi_{Vk}(\phi,\mathbf{0})$ weight function parts up to a $2k\pi/N$ rotation, whatever this function is, in the limit of the necessary rotation invariance. Therefore, the integral over the O unit circle can be written as an integral over a $\Delta \mathrm{O}_{(N)}$ arc, for sums of N operators all giving $\mathbf{t}^{\mathrm{cfib}}$, say

$$\mathbf{t}^V(\mathbf{0}) = N \int_{\Delta \mathrm{O}_{(N)}} \left(\frac{1}{N} \sum_{k=1}^{N} \mathbf{t}^P\left(\phi + \frac{2k\pi}{N}\right)\right)\psi_V(\phi,\mathbf{0})\mathrm{d}\phi = \mathbf{t}^{\mathrm{cfib}}\left(N\int_{\Delta \mathrm{O}_{(N)}} \psi_V(\phi,\mathbf{0})\mathrm{d}\phi\right) = \mathbf{t}^{\mathrm{cfib}}. \tag{16}$$

For any other $\mathbf{r}$ point in V the $\Delta_{\mathbf{r}}^k \mathrm{O}$ arcs are not identical. However, the $\widetilde{S}$—property remains fulfilled on the average over V, since the $\mathbf{r}$ interior points can be collected in N-sets of homologous points (Fig. 2) with regard to the rotation group letting V shape-invariant. Define $\Delta V = V/N$ as the ensemble made of one $\mathbf{r}_h$ point of each $((\mathbf{r}_1,\ldots,\mathbf{r}_N)$ N-point set, no matter how). This yields

$$\begin{aligned}\overline{\mathbf{t}^V} &= \frac{1}{N\Delta V}\int_{\Delta V} \sum_{h=1}^{N}\left(\sum_{k=1}^{N}\int_{\Delta_{\mathbf{r}}^{kh}\mathrm{O}} \mathbf{t}^P(\phi)\psi_{Vk}(\phi,\mathbf{r}_h)\mathrm{d}\phi\right)\mathrm{d}\mathbf{r} \\ &= \frac{1}{\Delta V}\int_{\Delta V}\left(\frac{1}{N}\sum_{k=1}^{N}\mathbf{t}^P\left(\phi+\frac{2k\pi}{N}\right)\right)\left(\sum_{h=1}^{N}\int_{\Delta_{\mathbf{r}}^h \mathrm{O}}\psi_V(\phi,\mathbf{r}_h)\mathrm{d}\phi\right)\mathrm{d}\mathbf{r} = \mathbf{t}^{\mathrm{cfib}},\end{aligned} \tag{17}$$

since the N-sums of the $\Delta_{\mathbf{r}}^{kh}\mathrm{O}$ arcs relative to each k boundary element for a (h) N-point set are so all made equal to the entire O unit circle. For $N = 2$ or 4, or for $N = 2$, according to the $\mathbf{C}$ tensor rank, the N elementary operator sum in Eq. (16) for a given ϕ still depends on ϕ, and therefore cannot be extracted from the weight function integral. So it is as well for the mean value in Eq. (17), and consequently $\mathbf{t}^V(\mathbf{0}) \neq \overline{\mathbf{t}^V} \neq \mathbf{t}^{\mathrm{cfib}}$. More generally, since replacing the linear boundary elements of a regular N-polygon by elements of any other shape, provided all elements are taken identical, still creates a fibre cross section which preserves the $\widetilde{S}$—property, the more or less exotic examples of Fig. 2 can be infinitely multiplied. Section 4 will discuss some particular ones.

3.2. 3D inclusions

Similar equations to Eqs. (15)–(17) formally apply to particular 3D "piece-wise regular" V inclusions, the boundary of which is characterized by N identical surface boundary elements which can be mapped on each other by "some rotation" around a O symmetry center. Consider then double $\boldsymbol{\omega} = (\theta, \phi)$ integrals over the $\Delta^k_{\mathbf{r}}\Omega$ solid angles in the unit sphere around a $\mathbf{r}$ interior point of V, which intercept the N surface boundary elements. From the O inclusion symmetry center, the $\Delta^k_{\mathbf{0}}\Omega$ solid angles are, up to a rotation, identical to a same $\Delta\Omega_{(N)}$ one. The boundary elements of such N-rotation invariant 3D shapes furthermore have a n-fold rotation symmetry axis, through O, which are n-symmetry axes for the inclusion. The $\Delta\Omega_{(N)}$ solid angle consequently shares into n identical $\Delta\Omega_{(nN)}$ subparts individually enough to define, by appropriate rotations, $n \times N$ homologous elements [5] of the V surface to be characterized by a same weight function. These rotations around O define, for each direction in $\Delta\Omega_{(nN)}$, $n \times N$ homologous directions through (or points on) the boundary of V. The 3D equivalent to Eqs. (15) and (16) at $\mathbf{r} = 0$ then can be written

$$\mathbf{t}^V(\mathbf{0}) = \sum_{k=1}^{N} \sum_{g(k)=1}^{n} \int_{\Delta_{\mathbf{0}}^{g(k)}\Omega} \mathbf{t}^P(\theta_{g(k)}, \phi_{g(k)})\psi_V(\theta_{g(k)}, \phi_{g(k)}, \mathbf{0})\mathrm{d}\omega_{g(k)},$$
$$= nN \int_{\Delta\Omega_{(nN)}} \left(\frac{1}{nN} \sum_{k=1}^{N} \sum_{g(k)=1}^{n} \mathbf{t}^P\left(\theta_{g(k)}, \phi_{g(k)}\right) \right) \psi_V(\omega, \mathbf{0})\mathrm{d}\omega. \tag{18}$$

One here shows that particular $n \times N$ sums of $\mathbf{t}^P(\theta_{g(k)}, \phi_{g(k)})$ elementary operators such as in Eq. (18) equal the $\mathbf{t}^{\mathrm{sph}}$ operator, independently on the $\boldsymbol{\omega}$-direction they correspond to in the $\Delta\Omega_{(nN)}$ solid angle which intercepts the reference surface boundary element of some V inclusion(s). For such V shapes, this will yield. $\mathbf{t}^V(\mathbf{0}) = \mathbf{t}^{\mathrm{sph}}$ since $nN\int_{\Delta\Omega_{(nN)}} \psi_V(\omega, 0)\mathrm{d}\omega = 1$, $\forall\ \psi_V(\omega, \mathbf{0})$ over $\Delta\Omega_{(nN)}$. Then, the 3D equivalent to Eq. (17) for the calculation of the $\mathbf{t}^V$ mean operator will follow, in similarly collecting the $\mathbf{r}$ interior points of V by sets of such $n \times N$ homologous, as above defined, positions around O. Existing such elementary operator sets constitute a sphere class of inclusion shapes with regard to the "$\widetilde{S}$—property". This existence is shown for the four-rank elasticity tensor first and for the second-rank tensors incidentally.

It is first recalled how the $\mathbf{t}^{\mathrm{sph}}$ Green operator integral for the sphere is obtained from the ϕ and $\cos\theta$ double integration of the non-zero elementary operator terms in Table 1. A first isotropic ϕ integration at fixed θ value yields the intermediate operator given Table 3. This operator is characteristic of any z-axially symmetric (not necessarily centrally symmetric) [6] inclusion, in the sense of a $\frac{2k\pi}{n}$ rotation invariance in ϕ around the symmetry axis ($n \neq 2, 4$, or $n \neq 2$ according to the $\mathbf{C}$ property tensor rank). The sphere operator then results from integrals of the form $\frac{1}{2}\int_{\theta=0}^{\pi} \cos^{2p}\theta \sin^{2q}\theta \sin\theta\, \mathrm{d}\theta$, with $p, q = 0, 1, 2$ and $p + q = 1, 2$, say, with $u = \cos(\theta) = \frac{1}{\sqrt{1+tg^2\theta}}$ and $H^{(p,q)}(u) = u^{2p}(1 - u^2)^q$:

$$\int_0^1 H^{(p,q)}(u)\mathrm{d}u = \begin{cases} 2/3 \text{ and } 1/3 \text{ for } (p,q) = (0,1) \text{ and } (1,0) \text{ resp.,} \\ 8/15, 3/15 \text{ and } 2/15 \text{ for } (p,q) = (0,2), (2,0) \text{ and } (1,1) \text{ resp.} \end{cases} \tag{19}$$

Only two over the five functions of Eq. (19) are independent (same relations as between the ϕ integrals in Eq. (12). Substituting these integrals to the θ functions in the elements of Table 3 yields the 3 non-zero sphere operator terms in elasticity

[5] This number reduces to $n \times N/2$ when the surface boundary elements of V are pair-wise opposite.

[6] The intermediate operator is at first characteristic of all z-axially symmetric, centrally reflected V and $-V$ body pairs. If V is also centrally symmetric, what furthermore requires the $z = 0$ plane as symmetry (mirror) plane, one has $V = -V$.

Table 3
Non zero terms of the z-axi-symmetric intermediate operator resulting from the ϕ isotropic integration of the elementary operators

	11	22	33	13	23	12
11	$(3/8)As^4\theta$ $(1/2)Bs^2\theta$	$(1/8)As^4\theta$ 0	$(1/2)As^2\theta c^2\theta$ 0			
22	$(1/8)As^4\theta 0$ 0	$(3/8)As^4\theta$ $(1/2)Bs^2\theta$	$(1/2)As^2\theta c^2\theta$ 0			
	$(1/2)As^2\theta c^2\theta$ 0	$(1/2)As^2\theta c^2\theta$ 0	$\mathrm{A}c^4\theta$ $\mathrm{B}c^2\theta$			
32				$(1/2)As^2\theta c^2\theta$ $B((s^2\theta/2)+c^2\theta)/4$		
31					$(1/2)As^2\theta c^2\theta$ $B((s^2\theta/2)+c^2\theta)/4$	
12						$(1/8)As^4\theta$ $Bs^2\theta/4$

$$t_{iiii}^{\mathrm{sph}} = \frac{3A+5B}{15}; \quad t_{(i,j),(i,j)}^{\mathrm{sph}} = \frac{2A+5B}{30}; \quad t_{ii,jj}^{\mathrm{sph}} = \frac{A}{15}.$$

Only two of these terms are independent, with the relation $t_{iiii}^{\mathrm{sph}} - t_{ii,jj}^{\mathrm{sph}} = 2t_{(i,j),(i,j)}^{\mathrm{sph}}$, as it also results from the analytically known $\mathbf{E}^{\mathrm{sph}}$ Eshelby operator with $\mathbf{t}^V = E^V{:}\mathbf{C}^{-1}$ (see Mura, 1987 for example). For rank-two $\mathbf{C}$ tensors, only the first line of Eq. (19) applies, for a single independent function. This yields the single term $t_{ii}^{\mathrm{sph}} = \frac{D}{3}$. As for the 2D case, the existence of V inclusions having a mean and a central Green operator integral equal to the sphere operator corresponds to the existence of $n \times N$-sets of $(\cos(\theta_{k,i}), \phi_{k,i})$ orientations for which the double (θ, ϕ) sums for the operator terms of Table 1, equal the above $(\cos(\theta), \phi)$ double integrals.

The simplest and more obvious shape cases corresponding to a space partitioning into $n \times N$ identical solid angles are regular N-polyhedrons, with $2N$ (or N) regular n-polygonal faces. Fig. 3 represents two such regular N-polyhedrons, i.e. the dodecahedron ($N = 12$) and the icosahedron ($N = 20$), reciprocal from each other as defined in Section 2.1, anticipating that the dodecahedral/icosahedral symmetry is a fundamental

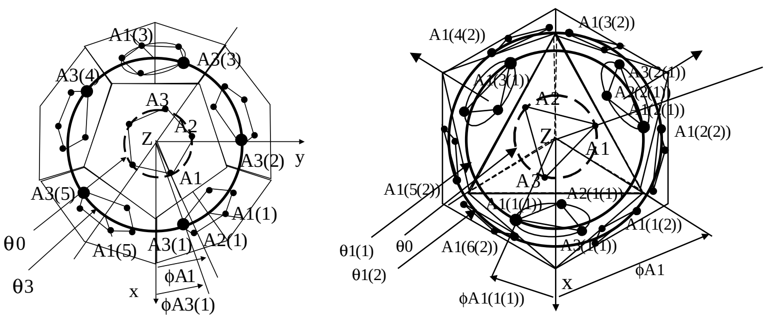

Fig. 3. Sets of homologous points on the faces of a regular dodecahedron (left) and icosahedron (right), and ϕ-symmetric groups of same θ orientation (black dots on same circles) with regard to a z symmetry axis.

one with regard to the here concerned $\widetilde{S}$—property. Their geometrical characteristics are recalled in Appendix B which then investigates, for all regular N-polyhedrons from $N = 4$ to $N = 20$, whether the particular "regular N-polyhedral" direction sets of the face normal (or the vertices for the reciprocal body) provide elementary operator sums equal to $\mathbf{t}^{\mathrm{sph}}$. For either four-rank or second-rank $\mathbf{C}$ tensors respectively, the regular N-polyhedral sets which fulfill this preliminary test selects the polyhedral V inclusion shape types which can fulfill the $\widetilde{S}$—property, provided that elementary operator sums over any other set of $n \times N$ homologous directions (or V surface points) in $\Delta\Omega_{(\mathrm{nN})}$ as previously defined, also equal $\mathbf{t}^{\mathrm{sph}}$. As shown in Appendix B, only the dodecahedral and the icosahedral direction sets fulfill the preliminary test for four-rank tensors, and the proof that, for these shapes, all the operator sums over the other $n \times N$ homologous direction sets also equal $\mathbf{t}^{\mathrm{sph}}$ is reported in Appendix B as well. It makes use of the fact that the n-fold symmetry axes of each n-polygonal face of these shapes also are n-fold symmetry axes for the inclusion itself. As one can see in Fig. 3, when a face normal of the considered regular N-polyhedron is oriented along the z axis (i.e. at $\theta = 0$), P-sets of $n(k)$ identical faces, $k = 1, \ldots, P$, have identically θk-oriented normal around z, at $\phi(g(k)) = 2g(k)\pi/(n(k))$ angles, with $g(k) = 1, \ldots, n(k)$. All $n(k)$ values are equal or multiple to the n face side number. As far as $n \neq 2, 4$, (or $\neq 2$ for rank-2 properties), appropriate i(k) grouping of $A_{i(g(k))}$ homologous V surface points (or $\mathrm{OA}_{i(g(k))}$ directions) among the $n \times N$ ones will yield ϕ-independent partial operator sums equal to, for each characteristic $\theta_{i(k)}$ value, the intermediate operator given in Table 3. Such groups are exemplified for both shapes drawn in Fig. 3, by the set of larger black dots on concentric circles around z. They correspond to direction sets of $\phi i(g(k)) = \phi i(n(k)) + 2g(k)\pi/(n(k))$ relative orientations, $g(k) = 1, \ldots, n(k)$, at fixed $k = 1, \ldots, P$ values. Grouping the corresponding n-set of points on the face of normal z, at $\theta 0$ from z, which fulfils the ϕ symmetry alone, yields the Table 3 intermediate operator for the $\theta 0$ value. Thus, the elementary operator sums in Eq. (18) equal $\mathbf{t}^{\mathrm{sph}}$ if the sums over the intermediate operators, from Table 3, for the $u_0 = \cos(\theta_0)$ and $u_{i(k)} = \cos(\theta_{i(k)})$ values, equal the related $u = \cos(\theta)$ integrals of Eq. (19). This yields the conditions

$$\frac{2}{N}\left(H^{(p,q)}(u_0) + \sum_{k=1}^{P}\sum_{i(k)=1}^{n} H^{(p,q)}(u_{i(k)})\right) = \begin{cases} 2/3 \text{ or } 1/3 \text{ for } (p,q) = (0,1) \text{ or } (1,0) \\ 8/15, 3/15 \text{ or } 2/15 \text{ for } (p,q) = (0,2), (2,0) \text{ or } (1,1) \end{cases}, \tag{20}$$

where $H^{(p,q)}(u)$ stands for $u^{2p}(1-u^2)^q$. For elasticity, i.e. for fourth-rank $\mathbf{C}$ tensors, this is shown verified (see Appendix B) for "dodecahedral sets" of homologous platelet operators, as well as for "icosahedral sets" of homologous platelet operators. Both cases in fact correspond to same sets of $n \times N/2 = 5 \times 6 = 3 \times 10 = 30$ operators. Since the dodecahedral and icosahedral sets respectively correspond to the 12 pentagonal faces of a regular dodecahedron and to the 20 triangular faces of a regular icosahedron, the $\widetilde{S}$—property is therefore fulfilled by these two reciprocal polyhedrons, as numerically shown in Nozaki and Taya (2001). For the particular directions of the face normal of the dodecahedron and of the icosahedron, the sets of 30 operators of $\mathbf{t}^{\mathrm{sph}}$ sum reduce to 6 and 10 respectively. For second-rank tensors, the first of Eq. (20) is verified to hold (see Appendix B) for the five regular polyhedral sets of orientations $N = 4, 6, 8, 12, 20$, [7] and the sphere class of polyhedrons fulfilling the $\widetilde{S}$—property then is larger than for elasticity-like properties which excluded $N = 4, 6$ or 8. Note that the $N = 4, 6, 8$ cases are linked since the cube and the regular octahedron are reciprocal shapes, while both the regular tetrahedral and octahedral direction sets (the face normal) are the set of the four (1 1 1) diagonals of a cube. The octahedron also is the centrally symmetric convex body of same brightness function as the regular tetrahedron (Gardner,

[7] Corresponding to the only existing 5 regular "Platon" polyhedra.

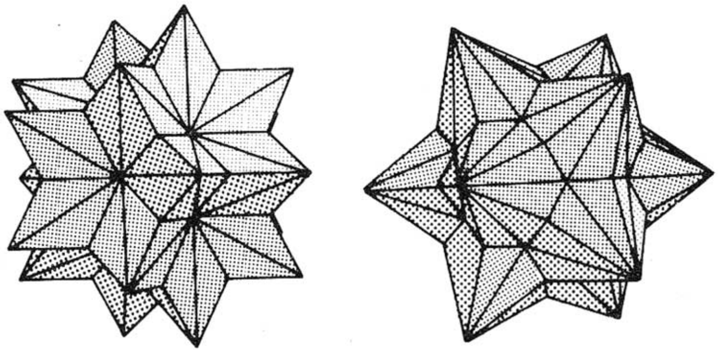

Fig. 4. Star shaped polyhedrons, from Janot (1992), both with 5-fold and 3-fold symmetry axes and derived from either the dodecahedron or the icosahedron.

1995). From all these results, the identity of the central and mean modified Green operator integrals with regard to some V domain, proves not due to an identity of the $\psi_V(\omega, \mathbf{0})$ central and $\overline{\psi_V}(\omega)$ mean weight function for V, but to an appropriate match between, on the one hand, the symmetry characteristics of the $\mathbf{C}$ tensor that represents the considered medium property and, on the other hand, those of the V shape, through the inverse Radon transform of its characteristic function. Therefore, as for fibre-like inclusions, the (3D) $\widetilde{S}$—property is also fulfilled, for the elasticity property, by all "regularly truncated" regular polyhedrons, all belonging to the icosahedral/dodecahedral family, examples of which are given in Kawashita and Nozaki (2001). In particular, the exemplified icosidodecahedron by these authors is a special regularly truncated dodecahedron, the reciprocal body of which (called triacontrahedron, see for example Janot, 1992) also is a special semi-regular (regularly truncated icosahedron) member of the sphere $\widetilde{S}$—class. This is because all these shapes have same sets of nN homologous points in the previously defined way. As stressed in Nozaki and Taya (2001), icosahedral symmetries are frequently observed in quasi-crystal structures. We here show on Fig. 4 typical structures, taken from Janot (1992), which are particular types of 3D star-shaped inclusions belonging to the sphere $\widetilde{S}$—class. They are basically built in replacing planar n-fold boundary elements of a dodecahedron ($n = 5$), or of its dual icosahedron ($n = 3$), by n-symmetrical pyramidal elements, turned towards the inside or the outside of the inclusion. For elasticity-like properties, all the so far identified polyhedrons of the sphere $\widetilde{S}$—class belong to the icosahedral/dodecahedral family, and other ones seem unlikely. Note that they are not necessarily convex. For rank-two $\mathbf{C}$ tensors, the sphere $\widetilde{S}$—class includes regularly truncated cubes, octahedrons and tetrahedrons, and non-convex polyhedrons of same symmetry as well. Furthermore, as for the fibre-like inclusions (but now remaining in the framework of isotropic symmetry for the material property), the $\widetilde{S}$—property can be immediately conclude to hold, without full calculation request of any weight function, for larger shape types of inclusions than polyhedrons. In particular, any piece-wise regular inclusion, convex or not, with vertices matching with the ones of a convex regular polyhedron of the sphere class, and with all identical boundary elements made of n homologous parts in radial symmetry, must still possess the $\widetilde{S}$—property. The request of a n-symmetry center for the boundary surface element makes the 3D symmetry condition for the $\widetilde{S}$—property stronger than the previously established one for the 2D case. The (d) boundary element of Fig. 2 fulfils Eqs. (16) and (17) without such central symmetry. However, excepted the d-like cases, many inclusion shapes which are kinds of 3D-equivalent to the 2D sections exemplified on Fig. 2 can be figured out as elements of the sphere $\widetilde{S}$—class. As for the fibre-like inclusions, the boundary elements of the shapes of the sphere class for the $\widetilde{S}$—property not needing to be planar, the inclusion shape types of the sphere $\widetilde{S}$—class is infinitely large. Particular ones are next commented.

4. Discussion

Compared to previous works stressing this $\widetilde{S}$—property, in elasticity framework, for polyhedrons of for polygonal fibres, the Radon transform method simply establishes that it holds for much larger types of shapes, for elasticity, and for even more shape types when considering material properties described by second-rank tensors. The Radon transform approach points that local or volume averages of modified Green operator integrals related to different V domain interiors can be identical although corresponding to different weight functions in the Radon inversion formula.

Now, while this $\widetilde{S}$—property appears widely fulfilled, i.e. for a large set of shape types, as far as one simultaneously has the material property of concern and the inclusion shape of same, fully or transversally, isotropic symmetry, no equivalent property seems preserved when either the material property symmetry or the inclusion shape symmetry is reduced, what makes, from the pessimistic viewpoint, the $\widetilde{S}$—property quite restricted. According to the problem of concern, it may be enough to consider inclusions only partly fulfilling the $\widetilde{S}$—property, what can enlarge the shape class of interest, or on the contrary it may be necessary to consider a sub class of inclusions fulfilling the $\widetilde{S}$—property plus additional ones. In order to open on further investigations, we here briefly address the following questions about existing inclusions: (1) partly fulfilling the $\widetilde{S}$—property for the mean Green operator integral only; (2) of "ellipsoidal classes", i.e. fully or partly fulfilling the $\widetilde{S}$—property with regard to ellipsoids; (3) of exactly same (i.e. transversally or fully isotropic) mean weight function as the sphere or cfib shape; (4) fulfilling the $\widetilde{S}$—property at more than one interior point, what becomes an $\widetilde{S}^{+}$—property, the limit of which is the "Ellipsoid property" of uniform interior modified Green operator integral.

4.1. Inclusions partly fulfilling the $\widetilde{S}$—property for the mean operator part

The $\widetilde{S}$—property associates the $\overline{S}$—property and the $S0$—property for the mean and for the central weight function respectively, say $\widetilde{S} = (\overline{S}, S0)$. The $\overline{S}$—property part is of specific interest for the many inclusion problems only involving the mean Green operator integral over bounded domains. The corresponding $\overline{S}$—class is larger than the $\widetilde{S}$—class, since it, at least, also includes multiply connected bounded domains, of which the centre may not be an interior point. Hollow inclusions or fibres belong to the $\overline{S}$—class if their (not necessarily identical) inner and outer boundaries share the same appropriate symmetry. 2D examples of hollow fibres of the $\overline{S}$—class result from associating two boundary types taken among those of Fig. 2, for a common minimal N symmetry. Hollow 3D inclusions of the $\overline{S}$—class can be obtained from similar associations of two shapes of the $\widetilde{S}$—class, having a nN symmetry in common. Patterns of identical

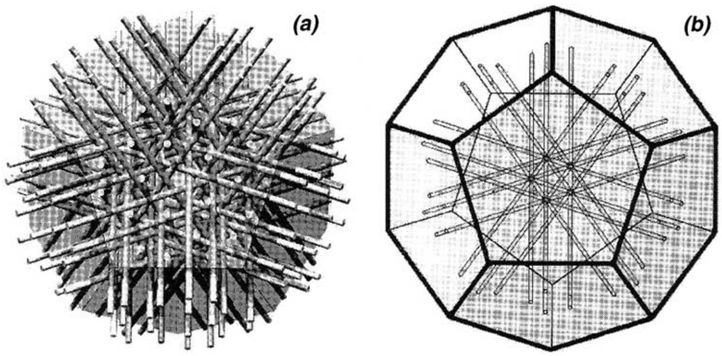

Fig. 5. Example of inclusion pattern of icosahedral symmetry, from Dendeviel et al. (2002). Left, global, right, detail of the fibre relative positions.

inclusions also belong to the $\overline{S}$—class when their spatial assemblage globally exhibits an appropriate (2D ou 3D) symmetry. In Fig. 2, taking by pairs the 2D exemplified $\Delta O_{(N)}$ shape elements, a N-set of same such pairs around O as drawn can represent either a hollow fibre, or a N-set of identical fibres which are $2k\pi/N$ rotated and assembled at contact. Moving them away from the figure centre, preserving the N-symmetry of their arrangement, creates a disjoint set of N fibres which is a 2D pattern of the $\overline{S}$—class. An interesting 3D example of inclusion pattern of the $\overline{S}$—class is illustrated on Fig. 5, which is issued from the work of Dendeviel et al. (2002). The authors have shown how to realise, and have realised, fibre patterns of icosahedral/dodecahedral symmetry embedded in resin, and have proposed a demonstration of expected isotropic elastic behaviour for so reinforced matrices, which unfortunately could not be revealed by mechanical testing on samples of necessarily cubic shapes. This 3D example has no 2D equivalent. Clearly here, from the Radon transform approach, reinforced matrices with isotropic arrangements of any inclusional domain type of the sphere $\overline{S}$—class will keep an isotropic behaviour for the considered property, because of isotropic mean Green operator integral for the inclusion, and the same hold for 2D problems with transversally isotropic symmetry.

4.2. Ellipsoidal shape classes for the full or partial $\widetilde{S}$—property

We here question whether a linear transform on some of the shapes of the $\widetilde{S}$—classes preserve (all or a part of) their $\widetilde{S}$—property, so constituting "ellipsoidal ($\widetilde{S}$, $\overline{S}$ or $S0$) classes" of shapes. The Appendix C recalls how, from the Radon transform approach, the modified Green operator integral of an elliptic fibre and of a spheroid are respectively derived from the the cylindrical fibre and the sphere ones, by linear transform. It is shown that, denoting $V \xrightarrow{\rho} V^{(\rho)}$ a general linear transform applied to some V shape, one has the two equivalent expressions

$$\mathbf{t}^{V(\rho)}(\mathbf{r}) = \int_{\Omega} \mathbf{t}^{P}(\omega)\psi_{V(\rho)}(\omega,\mathbf{r})\mathrm{d}\omega = \int_{\Omega} \mathbf{t}^{P(\rho)}(\omega)\psi_{V}(\omega,\mathbf{r})\mathrm{d}\omega \tag{21}$$

whether one transforms the weight function as $\psi_V(\omega,\mathbf{r}) \xrightarrow{\rho} \psi_{V(\rho)}(\omega,\mathbf{r})$, or the elementary operators as $\mathbf{t}^{P}(\omega) \xrightarrow{\rho} \mathbf{t}^{P(\rho)}(\omega)$, in the appropriate dual manners. Then, collecting N-sets of homologous elementary operators relative to a shape of the $\widetilde{S}$—classes will no more correspond, on the $V(\rho)$ stretched shape, to equal weights. Consequently these weights will no more be independent on the inclusion shape. Conversely, sets of transformed operators related to orientations kept in equal weights with regard to V, are no more of equal sum. It is enough to verify this for a degenerate set of face normal or vertex, taking the simplest case of a fibre of regular triangular cross section. As a result of the Radon approach, obvious exceptions appear to be inclusions, if any, of the sphere or cfib $\widetilde{S}$—classes having mean and/or central weight functions strictly equal to the $\psi_{\mathrm{sph}}(\omega)$ or $\psi_{\mathrm{cfib}}(\omega)$ weight functions.

4.3. Inclusion in the $\widetilde{S}$—class of same mean weight function as their reference shape

The proposed answer to this question motivated by the preceding conclusion uses the fact that (Franciosi and Lormand, 2004) the inverse Radon transform of $\overline{\psi_V}(\omega)$ is, in R^3 as in R^2, linked to the global covariance function of V at the origin of the covariance space, since, from Eq. (5b):

$$\int_V X_V(\mathbf{r})\mathrm{d}\mathbf{r} = \int X_V(\mathbf{r})X_V(\mathbf{r})\mathrm{d}\mathbf{r} = \lim_{\mathbf{r}0\to 0}\left(\int X_V(\mathbf{r})X_V(\mathbf{r}+\mathbf{r}0)\mathrm{d}\mathbf{r}\right) = v = C^V(\mathbf{r}0 = 0). \tag{22}$$

This covariance function (or covariogram) at origin does not, alone, uniquely define a V domain (take the example of two non-centrally symmetric bodies, which make a V, $-V$ centrally reflected pair), and so it is for $\overline{\psi_V}(\omega)$. The fundamental descriptor of a V shape anisotropy is the $C^V(0)_{,|\mathbf{r}0|\omega}$ derivative of $C^V(\mathbf{r}0)$ at

$\mathbf{r}0 = 0$, which equals minus the total projected area of V in the $\boldsymbol{\omega}$ direction, say $-A_V(\omega)$ (Serra, 1982), as shown in Fig. 1. This definition does not only hold for convex bodies, but if V is a convex body, $A_V(\omega)$ is the $B_V(\omega)$ brightness of V. Most available work concerns convex V bodies, and from Eq. (22), using Eq. (5b) [8], one then obtains for this derivative

$$\begin{aligned} C^V(0)_{,|\mathbf{r}0|\boldsymbol{\omega}i} &= -\frac{1}{8\pi^2}\int_\Omega\left(\int_{D_V^-(\omega)}^{D_V^+(\omega)} s''_V(z,\omega)s'_V(z,\omega)\mathrm{d}z\right)|\cos(\omega i,\omega)|\,\mathrm{d}\omega \\ &= -\frac{1}{8\pi^2}\int_\Omega (s'_V(D_V(\omega))^2|\cos(\omega i,\omega)|\,\mathrm{d}\omega = -\frac{4\pi^2}{8\pi^2}\int_\Omega (R_1^V(\omega)R_2^V(\omega))|\cos(\omega i,\omega)|\,\mathrm{d}\omega \\ &= -B_V(\omega i) = -A_V(\omega i). \end{aligned} \tag{23}$$

The $C^V(0)_{,|\mathbf{r}0|\boldsymbol{\omega}}$ derivatives define, in $R^n \times R$, a tangential cône to $C^V(\mathbf{r}0)$ at $\mathbf{r}0 = 0$. Cuts of this cône by any parallel plane to its (convex and centrally symmetric) basis define the shape of the $C^V(\mathbf{r}0)$ isovalues near $\mathbf{r}0 = 0$ and cuts by cylinders of vanishing radius r_0 define the covariance isocontours near origin. Thus, as far as bodies of isotropic $\overline{\psi_V}(\omega)$ mean weight function identify to bodies of isotropic covariance at origin, they will necessarily belong to the class of bodies with constant $A_V(\omega)$ total projected area, i.e. of constant $B_V(\omega)$ brightness for the convex ones, which, per se, constitutes a remarkable shape class. Examples of convex bodies of constant brightness, in R^3 and in R^2, not congruent to a sphere or to a disc respectively, are given in Gardner (1995). We here report in Appendix D examples (not found elsewhere) of non-convex 2D and 3D bodies of constant $A_V(\omega)$ total projected area, also belonging to the $\widetilde{S}$—class here of concern.

4.4. The $\widetilde{\mathbf{S}}^+$—property of same Green operator integral at more than one interior point

Rephrased according to the connexion with covariograms, the inclusions fulfilling the $\widetilde{S}$—property would be those having the same covariogram at origin (of the covariance space) as a reference shape. This covariogram also defines the weight function at one interior point, generally a symmetry centre, of the inclusion. A reference shape of uniform weight function is then a shape whose covariogram at origin also defines the weight function at all interior points. This is an other definition for the "Ellipsoid property", so far only ensured fulfilled by ellipsoids, of a uniform interior modified Green operator integral. Before considering the existence of non-ellipsoidal inclusions fulfilling this Ellipsoid property, lets question the existence of inclusions fulfilling the intermediate property, denoted $\widetilde{S}^+$—property, of a covariogram at origin also defining their weight function at more than one interior point. Existence of such non-ellipsoidal inclusions remains unexpected, mostly from the fact that in Eq. (15) for fibres, the $\Delta_\mathbf{r}^k O$ arcs are not identical for any $\mathbf{r}$ point in V but the centre, with a similar remark holding for 3D inclusions, for nN sets $\Delta_\mathbf{r}^{g(k)}\Omega$ solid angles, viewed from $\mathbf{r} \neq \mathbf{0}$. However, there is no impossibility proof resulting straightforwardly from the present Radon approach. So it is as well with regard to the Ellipsoid property.

Now, N-polygonal or N-polyhedral V shapes which are close to respectively a cylindrical fibre and a sphere, have a modified Green operator integral not far from uniformity from one interior point to another, and the larger is N, the weaker is the operator inhomogeneity in V. While the operator differences between a sphere and a dodecahedron for example are weak and not of much interest, the operator interior inhomogeneity may be much more important for other inclusions of the cfib or sphere $\widetilde{S}$—class with highly non-linear or non-planar boundary element respectively. A same mean value can therefore be associated with many inhomogeneous, shape-dependent, interior operator fields. At last, it has been conversely seen from

[8] The last equality is the definition of $A_V(\omega)$, and the previous one comes from the V section expression near the M point of the V surface of $\boldsymbol{\omega}$-oriented normal, in terms of the main curvature radii at M, $s_V(z,\omega) = 2\pi(R_1(\omega)R_2(\omega))^{1/2}(D_V\omega) - z)$. For ellipsoids, $R_1^V(\omega)R_2^V(\omega) = (3v/4\pi D_V(\omega)^2)^2$.

Table 2 that there are several shape-invariant combinations of terms of the $\mathbf{t}^P(\omega)$ elementary operator (see Eqs. (10b) and (11b)), such as its trace which is equal to the D constant for the rank-2 tensors, and to the $A+2B$ constant for the rank-4 ones. Thus, Eq. (2) yields combinations of terms of $\mathbf{t}^V(\mathbf{r})$ which are uniform over V, whatever its shape is. For the trace example

$$\mathrm{tr}(\mathbf{t}^V(\mathbf{r})) = \int_\Omega \mathrm{tr}(\mathbf{t}^P(\omega))\psi_V(\omega,\mathbf{r})\mathrm{d}\omega = \begin{cases} (A+2B) \text{ for } t^P_{pqjn}(\omega), \\ D \text{ for } t^P_{qn}(\omega). \end{cases} \tag{24}$$

This is due to $M(\omega)$ operators in Eq. (3b) which are 3D or 2D $\boldsymbol{\omega}$-independent when the material property is of fully or at least transversally (and fibre oriented) isotropic symmetry, respectively.

5. Conclusion

From the use of the Radon transform method in the analysis of the modified Green operator integral inside inclusions, one has investigated the shapes having the here called $\widetilde{S}$—property. That is 3D inclusions (resp. 2D fibres) having a mean and a central modified Green operator integral both equal to the uniform one of the sphere (resp. the fibre of circular cross section). The so obtained "sphere class" and "cylindrical fibre class" with regard to this $\widetilde{S}$—property, have been shown, for fully (resp. transversally) isotropic elasticity of the embedding matrix, and without full calculations of shape characteristic weight functions, to not be only constituted of regular convex polyhedral inclusions (resp. polygonal fibres). They have been shown to also contain any convex or non-convex shape having the same fundamental symmetry as one of such regular polytope of the class. Such shape classes for the $\widetilde{S}$—property are established for the fourth-rank elasticity property as well as for second-rank physical properties, with larger shape classes in the latter case. The partial $\overline{S}$—property for only the mean operator value concerns even larger classes of bounded domains, including hollow inclusions or multiply connected bounded domains, such as inclusions patterns, provided they also fulfil the appropriate symmetry. The conditions under which the $\widetilde{S}$—property may be, totally or partly preserved, for elements of the sphere or of the cylindrical fibre class, if any, by linear transform, so constituting ellipsoidal shape classes, have been discussed. Among candidates are shapes, if any, having the same (3D or 2D isotropic) weight function as their reference shape. Considering only the partial $\overline{S}$—property, examples of 2D and 3D inclusions fulfilling the condition of constant total projection area function, a necessary condition to have an isotropic mean weight function in R^3 or in R^2, are given. The Radon transform method neither has provided evidence of non-ellipsoidal inclusions fulfilling the "Ellipsoid property" of a uniform modified Green operator integral at any interior point, nor has decisively refuted this possibility. Uniformity of several invariant characteristics of the modified Green operator integral, such as its trace, proves to hold over any 3D or fibre-like inclusion for respectively fully and transversally (fibre-oriented) isotropic material property.

Appendix A. The Radon transform and its inversion formula for $X_V(\mathbf{r})$ in $\mathbf{R}^2$

The $X_V(\mathbf{r})$ characteristic function of some V domain in R^2 (Gel'fand et al., 1966), in comparison to Eq. (5a) for V in R^3, and with $1_V(z',\phi)$ its Radon transform, i.e. the length of the chord intercepted by V on the line $z' = \mathbf{r}\cdot\boldsymbol{\omega}$, $\boldsymbol{\omega} = (\sin(\phi), \cos(\phi))$, writes in terms of the Radon inversion formula:

$$X_V(\mathbf{r}) = -\frac{1}{4\pi^2}\int_{\mathrm{O}}\left(\int_{D_V^-(\phi)}^{D_V^+(\phi)}\left(\frac{l_V(z',\phi)\mathrm{d}z'}{(z-z')^2}\right)\right)\mathrm{d}\phi = \frac{1}{4\pi^2}\int_{\mathrm{O}}\left(\int_{D_V^-(\phi)}^{D_V^+(\phi)}\left(\frac{l'_V(z',\phi)\mathrm{d}z'}{z-z'}\right)\right)\mathrm{d}\phi = 1. \tag{A.1a}$$

In R^2 as in all even n spaces, the inverse Radon transform of $X_V(\mathbf{r})$ for any $\mathbf{r}$ point interior to V, does not only involve the (n-1) planes (lines for R^2) intersecting V that contain $\mathbf{r}$, but all of them. A resolution of Eq. (A.1a) in terms of ordinary integrals is given in Natterer (1986), yielding with $\zeta = z' - z$

$$X_V(\mathbf{r}) = -\frac{1}{4\pi^2}\int_O\left(\int_{D_V^-(\phi)-z}^{D_V^+(\phi)-z}\left(\frac{1}{\zeta}\frac{\mathrm{d}}{\mathrm{d}z}(l_V(z+\zeta,\phi))\right)\mathrm{d}\zeta\right)\mathrm{d}\phi = \int_O \psi_V(\phi,\mathbf{r})\mathrm{d}\phi = 1. \tag{A.1b}$$

For a V disc of radius R, $l_V(z+\zeta,\phi) = 2\sqrt{R^2-(z+\zeta)^2}\ \forall\ \phi \in (0,2\pi)$. In Eq. (A.1b), the ζ integral over $[-R-z, R-z]$ can be set, $\forall\ z$, equal to $2\int_{\lambda=-1}^{\lambda=1}\frac{\mathrm{d}\lambda}{\lambda\sqrt{\lambda^2-1}} = -2\pi$, $\lambda = R/(z+\zeta)$. Thus, as for the cylindrical fibre in R^3 space Section 2.1, $\psi_{\text{disc}}(\phi,\mathbf{r}) = \psi_{\text{disc}}(\phi) = \psi_{\text{cfib}}(\frac{\pi}{2},\phi) = \frac{1}{2\pi}$ uniformly.

Appendix B. Dodecahedron, icosahedron and other main sphere class elements for the $\widetilde{\mathrm{S}}$—property

B.1. Geometry

The regular dodecahedron and regular icosahedron have reciprocal characteristics (12 5-fold faces/vertices, 20 3-fold vertices/faces, resp.). The Fig. B1 shows a half section of an icosahedron through a 5-fold EA//z symmetry axis and one AB edge, joining the OA = OB = R vertices. This plane contains two face normal OC = OD = h. $(\mathrm{A}\widehat{\mathrm{O}}\mathrm{B}) = 2\gamma = \mathrm{arctg}(2)$, $(\mathrm{C}\widehat{\mathrm{O}}\mathrm{D}) = \lambda = \arcsin(2/3)$, $(\mathrm{B}\widehat{\mathrm{O}}\mathrm{C}) = (\mathrm{D}\widehat{\mathrm{O}}\mathrm{E}) = \eta = (\pi - \lambda - 2\gamma)/2$. Thus, ϕi angles ($i = 0,1,2$) around z are

$$\phi i = \frac{2i}{5}\pi, \quad \theta 1 = \eta, \quad \theta 2 = \eta + \lambda; \quad \phi i = \frac{2i+1}{5}\pi, \quad \theta'^1 = \pi - \eta, \quad \theta'^2 = \pi - (\eta+\lambda), \quad i = 0,1,2. \tag{B.1}$$

Taking an icosahedron face normal in the z direction amounts to set $\theta = \theta 1 = 0$ along OC or OD. It thus comes for the other face normal, of $\theta 2$, $\theta 3$ angles with z, the related $\phi i(k)$ angles

$$\phi i(2) = \frac{2i\pi}{3}, \quad \theta 2 = \lambda, \quad \theta' 2 = \pi - \lambda; \quad \phi i(3) = \frac{2i\pi}{3} \pm \frac{\pi}{6}, \quad \theta 3 = \arccos\left(\frac{1}{3}\right), \quad \theta' 3 = \pi - \theta 3, \quad i = 0,1,2. \tag{B.2}$$

For the dodecahedron, 2γ is the angle between two neighboring face normal, and the same solutions hold when OA, OE and OB are face normal, and OC, OD are vertices.

B.2. Face normal and vertex sets

Consider regular (i.e. equally weighted and defining identical $\Delta\Omega_{(N)}$ solid angles) polyhedral N-sets of directions from some O origin, fulfilling a n-fold rotation symmetry around the z axis which may or not carry one direction of the considered N-set. These two cases are denoted (F) and (V) for "face normal" and "vertex" z-symmetry axis respectively. Regular N-polyhedrons generally (but not necessarily) having pair-wise opposite faces, one can a priori write [9] $N = 2(1+\sum_{k=1}^{P} n(k))$ for (F), and $N = 2\sum_{k=1}^{P} n(k)$ for (V), with when $n(k) = n$ for all k, $N = 2(nP+1)$ and $N = 2nP$ respectively. The $(\theta = 0,\pi)$ face pair for (F) being specialized, for each k direction one associates a pair of $\theta k \neq 0$ and $\theta' k = \pi - \theta k$ angles to same $\cos^{2p}\theta(k)$ values, $p = 1,2$. Restrictions are $n(k) \neq 2,4$ or $n(k) \neq 2$ for fourth-rank or second-rank tensors

[9] For no pair-wise opposite faces, as the $N = 4$ regular tetrahedron a factor 2 is removed from N and related formulae.

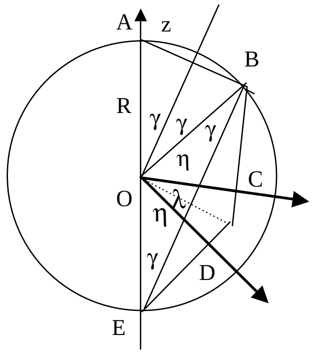

Fig. B1. Half section of a regular dodecahedron, through a 5-fold symmetry axis EA, and one of its five edges at A.

respectively. $N \leqslant 20$ will imply $P \leqslant 2$. Thus, from Eq. (20), N-sets of (F) or (V) directions fulfilling the sphere $\widetilde{S}$—property in elasticity must simultaneously solve the two independent equations for $q = 0$, $p = 1, 2$

Function	(F) z-symmetry axis	(V) z-symmetry axis
(1) $\mathrm{Cos}^2(\theta)$	$\frac{2}{N}\left(1 + \sum_{k=1}^{P} n(k)u_k^2\right) = \frac{1}{3}$	$\frac{2}{N}\left(\sum_{k=1}^{P} n(k)u_k^2\right) = \frac{1}{3}$
(2) $\mathrm{Cos}^4(\theta)$	$\frac{2}{N}\left(1 + \sum_{k=1}^{P} n(k)u_k^4\right) = \frac{1}{5}$	$\frac{2}{N}\left(\sum_{k=1}^{P} n(k)u_k^4\right) = \frac{1}{5}$

- $P = 1$: Eqs. (F1) and (F2) are solved for $\theta 1 = \mathrm{arctg}(2) = 2\gamma$. The unique solution $n(1) = 5$ defines the $N = 2 \times 5 + 2 = 12$ dodecahedral set of directions; Eqs. (V1) and (V2) have no simultaneous solutions;
- $P = 2$: Eqs. (F1) and (F2) are solved for $\theta 1 = \mathrm{arcos}(\sqrt{5/3}) = \lambda$ and $\theta 2 = \mathrm{arcos}(1/3)$. The unique solution $n(1) = 3$ and $n(2) = 6$ yields the $N = 2 + 2 \times 3 + 2 \times 6 = 20$ icosahedral set (Eq. (B.2)); Eqs. (V1) and (V2) are solved for $(\theta 1, \theta 2) = \mathrm{arctg}(2(7 \pm 3\sqrt{5}))^{1/2} = (\eta, \eta + \lambda)$, $\forall\, n$, when $n = n(1) = n(2)$. This again defines, for $n = 3$, the dodecahedral set, $N = 3 \times 4 = 12$, and for $n = 5$, the icosahedral set, $N = 5 \times 4 = 20$ (Eq. (B.1)). For second-rank tensors, only the first F1 of V1 function for $\cos^2(\theta)$ needs to be fulfilled.
- $P = 1$: Eq. (F.1) has solutions for $\cos^2(\theta 1) = (n - 2)/3n$. $n = 3$ is the $\pm V_T$ tetrahedron ($N = 1 + 3n = 4$) with a z-vertex It also is the V_O octahedron ($N = 2 + 2n = 8$) with a z-oriented face normal pair, and other ones at $\theta 1 = \mathrm{arcos}(\pm 1/3)$); Eq.(V.1) is fulfilled for $\cos\theta 1 = 1/\sqrt{3}$, $\forall\, n$. For $n = 3$, $N = 2n = 6$ is the V_C cube with a z-oriented (1 1 1) diagonal (vertex). It is also the rhombo-dodecahedron (12 identical lozenges), when adding 2×3 faces at $\theta = \pi/2$, of zero contribution to Eq. (V.1). Furthermore, since $n = 4$ is allowed there, the same $\theta 1$ angle also corresponds to the V_O regular octahedron $N = 8$.

B.3. Homologous nN sets

For both the dodecahedron and the icosahedron of Fig. 3, with one face pair of normal z, $S0$ say, any other face is obtained by a (ϕ) rotation around the z axis followed by a (θ) rotation around a normal direction to z, the (θ) angle between the two face normal having an equal θk value for all faces of a same $k = 1, \ldots, P$ subgroup as previously defined. The same transformations map any n-set of homologous points of $S0$ on their homologous n-sets of the other faces. We abbreviate "sin, cos" as "s, c" respectively.

For each k value, the first, ϕ_k, rotation around z transforms the $\overrightarrow{\omega 0}_i = (s\theta 0 c\phi_i, s\theta 0 s\phi_i, c\theta 0)$ directions into the $\overrightarrow{\omega 0}_{ik} = (s\theta 0 c\phi_{ik}, s\theta 0 s\phi_{ik}, c\theta 0)$ ones, where $\phi_{ik} = \phi_k + \phi_i = \phi_k + \phi 0 + 2i\pi/n = \phi 0_k + 2i\pi/n$. The second $(g(k))$-rotations of $R^{g(k)}$ matrices, for a same θk angle, are around the $d^{g(k)} = (d_1^{g(k)}, d_2^{g(k)}, 0) = (-s(\phi_k^* + 2g(k)\pi/n(k)), c(\phi_k^* + 2g(k)\pi/n(k)), 0)$ directions, $g(k) = 1, \cdots, n(k)$. They yield the $(\overrightarrow{\omega 0}_{\mathrm{igk}}) = R^{g(k)}\,\overrightarrow{\omega 0}_{\mathrm{ik}}$ vectors, defined by $(\tilde{\theta}_{igk}, \tilde{\phi}_{igk})$ angles. As stressed in Section 3.2, ϕ-symmetry around the z axis is fulfilled by grouping the $n(k)$ directions of same $\tilde{\theta}_{igk}$ angle with z. The $\cos(\tilde{\theta}_{igk})$ functions then are given by the third (z) component of the $(\overrightarrow{\omega 0}_{\mathrm{igk}})$ vectors, which writes (Lequeu et al. (1987))

$$\begin{aligned}\cos(\tilde{\theta}_{igk}) &= c\theta 0 c\theta k + s\theta 0 s\theta k(d_2^{g(k)} c(\phi 0_k + 2i\pi/n) - d_1^{g(k)} s(\phi 0_k + 2i\pi/n)) \\ &= c\theta 0 c\theta k + c\left(\phi_k^* - \phi 0_k + \frac{2\pi}{n}\left(\frac{g}{l_k} - i\right)\right) s\theta 0 s\theta k.\end{aligned}$$

This provides the sums for $p = 1, 2$, with $l_k = (n(k)/n) \geqslant 1$ an integer

$$\sum_{k=1}^{P} \sum_{g(k)=1}^{n(k)} sum_{i(g(k))=1}^{n} (\cos(\tilde{\theta}_{igk}))^{2p} = \sum_{k=1}^{P} \sum_{g(k)=1}^{n(k)} \sum_{i(g(k))=1}^{n} \left(c\theta 0 c\theta k + c\left(\phi_k^* - \phi 0_k + \frac{2\pi}{n}\left(\frac{g}{l_k} - i\right)\right) s\theta 0 s\theta k\right)^{2p}.$$

With odd power terms vanishing, these sums respectively reduce for $p = 1$ and 2

$$p = 1 : n(c\theta 0)^2 \sum_{k=1}^{P} n(k)(c\theta k)^2 + (s\theta 0)^2 \sum_{k=1}^{P} (s\theta k)^2 \sum_{g(k)=1}^{n(k)} \sum_{i(g(k))=1}^{n} \left(c\left(\phi_k^* - \phi 0_k + \frac{2\pi}{n}\left(\frac{g}{l_k} - i\right)\right)\right)^2.$$

Inverting the (i) and (g) sums and using Eq. (14) for $(p, q) = (1, 0)$, reduces the (ϕ) sums to $(n/2)n(k)$. Adding $n(c\theta 0)^2$ for the $S0$ face contribution, and dividing by $nN/2$, Eq. (20) for $(p, q) = (1, 0)$ reads

$$\frac{2}{nN}\left(n \sum_{k=1}^{P} n(k)\left((c\theta 0)^2 (c\theta k)^2 + \frac{1}{2}(s\theta 0)^2 (s\theta k)^2\right) + n(c\theta 0)^2\right). \tag{B.3}$$

$$\begin{aligned}p = 2 : n(c\theta 0)^4 \sum_{k=1}^{P} n(k)(c\theta k)^4 &+ (s\theta 0)^4 \sum_{k=1}^{P} (s\theta k)^4 \sum_{g(k)=1}^{n(k)} \sum_{i(g(k))=1}^{n} \left(c\left(\phi_k^* - \phi 0_k + \frac{2\pi}{n}\left(\frac{g}{l_k} - i\right)\right)\right)^4 \\ &+ 6(c\theta 0 s\theta 0)^2 \sum_{k=1}^{P} (c\theta k s\theta k)^2 \sum_{g(k)=1}^{n(k)} \sum_{i(g(k))=1}^{n} \left(c\left(\phi_k^* - \phi 0' + \frac{2\pi}{n}\left(\frac{g}{l_k} - i\right)\right)\right)^2.\end{aligned}$$

Using Eq. (14) for (p, q)=(2, 0) reduces the (ϕ) sums of the second term to $(3n/8)n(k)$. Adding $n(c\theta 0)^4$ and dividing by $nN/2$ yields Eq. (20) for $(p, q) = (2, 0)$ as

$$\frac{2}{nN}\left(n \sum_{k=1}^{P} n(k)\left((c\theta 0)^4 (c\theta k)^4 + \frac{3}{8}(s\theta 0)^4 (s\theta k)^4 + 3(c\theta 0 s\theta 0)^2 (c\theta k s\theta k)^2\right) + n(c\theta 0)^4\right) \tag{B.4}$$

Owing to the ϕ symmetry around z, the sums in Eqs. (B.3) and (B.4) only depend on the $\theta 0$ selected n-set characteristic, and on the θk angles between the S0 face normal and the other ones.

For the dodecahedron, $P = 1$, $n(1) = n = 5$, $\cos\theta 1 = 1/\sqrt{5}$. This yields, $\forall\ \theta 0$

$$(1)\quad \frac{1}{6}\left((c\theta 0)^2\left(1 + 5\left(\frac{1}{\sqrt{5}}\right)^2\right) + (s\theta 0)^2\left(\frac{2}{\sqrt{5}}\right)^2 \frac{5}{2}\right) = \frac{1}{3},$$

$$(2)\ \ \frac{1}{6}\left((c\theta 0)^4\left(1+5\left(\frac{1}{\sqrt{5}}\right)^4\right)+(s\theta 0)^4\left(\frac{2}{\sqrt{5}}\right)^4\frac{15}{8}+15\left(\frac{2s\theta 0c\theta 0}{5}\right)^2\right)=\frac{1}{5}.$$

For the icosahedron, $P = 2$, $n(1) = 3\ \cos\theta 1 = \sqrt{5}/3$, $n(2) = 6$, $\cos\theta 2 = 1/3$. This yields, $\forall\ \theta 0$

$$(1)\ \ \frac{1}{10}\left((c\theta 0)^2\left(1+3\left(\frac{\sqrt{5}}{3}\right)^2+6\left(\frac{1}{3}\right)^2\right)+(s\theta 0)^2\left(\frac{3}{2}\left(\frac{2}{3}\right)^2+\frac{6}{2}\left(\frac{2\sqrt{2}}{3}\right)^2\right)\right)=\frac{1}{3},$$

$$(2)\ \ \frac{1}{10}\left((c\theta 0)^4\left(1+\frac{75}{81}+\frac{6}{81}\right)+(s\theta 0)^4\left(\frac{18}{81}+\frac{144}{81}\right)+6(c\theta 0s\theta 0)^2\left(\frac{30}{81}+\frac{24}{81}\right)\right)=\frac{1}{5}.$$

For rank-2 tensors, the $p = 1$ sum in Eq. (20) equals 1/3 as well for (a) the cube (for any 4-sets of homologous points on three orthogonal faces), and (b) the octahedron and the tetrahedron (for any 3-sets of homologous points on four faces oriented as the (1 1 1) cube diagonal or vertices):

$$(a)\ \ \frac{1}{3}\left((c\theta 0)^2+\sum_{i=1}^{2}\left(0+\cos\left(\phi 0+\frac{i\pi}{2}\right)s\theta 0\right)^2\right)=\frac{1}{3},$$

$$(b)\ \ \frac{1}{4}\left((c\theta 0)^2+3\left(\frac{c\theta 0}{3}\right)^2+\sum_{i=1}^{3}\cos\left(\phi 0+\frac{2i\pi}{3}\right)\left(\frac{2\sqrt{2}s\theta 0}{3}\right)^2\right)=\frac{1}{4}\left(\frac{4}{3}(c\theta 0)^2+\frac{3}{2}\cdot\frac{8}{9}(s\theta 0)^2\right)=\frac{1}{3}.$$

Appendix C. Linear transforms of the sphere and of the cylindrical fibre weight functions

The $\psi_{\rho\mathrm{fib}}(\alpha)$ weight function for the ρ-elliptic fibre of z axis, resulting from a ρ stretch of the cfib along the y direction ($\phi = \pi/2$), comes from the $tg\phi = \rho tg\alpha$ transform. Writting $\mathrm{d}\phi = \rho(\frac{1+tg^2\alpha}{1+\rho^2tg^2\alpha})\mathrm{d}\alpha$, gives $\psi_{\rho\mathrm{fib}}(\alpha) = \frac{\rho}{2\pi}(\frac{1+tg^2\alpha}{1+\rho^2tg^2\alpha})$ which, using $\cos^{2p}(\alpha) = \frac{1}{(1+tg^2(\alpha))^p}$, $p = 1,2$, and [10] $\mathrm{d}(tg\alpha) = (1 + tg^2\alpha)\mathrm{d}\alpha$, yields to fulfil the integral pair

$$\int_{0,2\pi}\frac{\rho}{2\pi}\left(\frac{1+tg^2\alpha}{1+\rho^2tg^2\alpha}\right)\frac{\mathrm{d}\alpha}{1+tg^2\alpha}=\frac{\rho}{1+\rho};\int_{0,2\pi}\frac{\rho}{2\pi}\left(\frac{1+tg^2\alpha}{1+\rho^2tg^2\alpha}\right)\frac{\mathrm{d}\alpha}{(1+tg^2\alpha)^2}=\frac{\rho(1+2\rho)}{2(1+\rho)^2}. \tag{C.1}$$

This results in the non-zero operator terms of Table C1, reducing to Eqs. (12) and (13) when $\rho = 1$. $\psi_{\rho\mathrm{fib}}(\alpha) \equiv \frac{1}{D^2(\alpha)} = R^{*^2}(\alpha)$, with $R^*(\alpha)$ the radius vector of the V^* reciprocal $\frac{1}{\rho}$-ellipse. $\mathbf{t}^{\rho\mathrm{fib}}$ agrees with the Eshelby tensor for this symmetry (Mura, 1987). When $\rho \to \infty$, $\mathbf{t}^{\rho fib} \to \mathbf{t}^P(\frac{\pi}{2}, 0)$ the platelet of x normal, when $\rho \to 0$, $\mathbf{t}^{\rho fib} \to \mathbf{t}^P(\frac{\pi}{2}, \frac{\pi}{2})$, the platelet of y normal.

The $\psi^{\rho\mathrm{sph}}(\beta,\alpha)$ weight function for the ρ-spheroid of z axis, resulting from a ρlinear transform of the sphere in the z direction ($\theta = 0$), comes from the transform $tg\theta = \frac{1}{\rho}tg\beta$, $\alpha = \phi$. This gives, $\forall\ \phi$, with $\sin(\theta)\mathrm{d}\theta = \frac{1}{\rho^2}((1 + tg^2\beta)/(1 + \frac{tg^2\beta}{\rho^2}))^{\frac{3}{2}}\sin(\beta)\mathrm{d}\beta$, and $u = \cos(\beta)$

$$\psi_{\rho sph}(\beta,\alpha)=\frac{1}{4\pi\rho^2}\left((1+tg^2\beta)\Big/\left(1+\frac{tg^2\beta}{\rho^2}\right)\right)^{\frac{3}{2}}=\frac{1}{4\pi\rho^2}\left(\frac{1}{1+(\rho^2-1)u^2}\right)^{\frac{3}{2}}.$$

[10] $\rho(1 + 2tg^2\alpha)\mathrm{d}\alpha = (1 + tg^2\phi)\mathrm{d}\phi \to \int_{0,2\pi}\frac{\rho}{2\pi}\left(\frac{1+tg^2\alpha}{(1+\rho^2tg^2\alpha)}\right)\frac{\mathrm{d}\alpha}{(1+tg^2\alpha)^p} = \frac{1}{2\pi}\int_{0,2\pi}\left(\frac{\rho^2}{\rho^2+tg^2\phi}\right)^p\mathrm{d}\phi,\ p = 1, 2.$

Table C1
The non-zero terms of the Green operator integral of a ρ-elliptic fibre

	11	22	33	23	31	12
11	$\frac{\rho(1+2\rho)A}{2(1+\rho)^2}+\frac{\rho B}{1+\rho}$	$\frac{\rho A}{2(1+\rho)^2}$	0			
22	$\frac{\rho A}{2(1+\rho)^2}$	$\frac{(2+\rho)A}{2(1+\rho)^2}+\frac{B}{1+\rho}$	0			
33	0	0	0			
23				$\frac{B}{4(1+\rho)}$		
31					$\frac{\rho B}{4(1+\rho)}$	
12						$\frac{\rho A}{2(1+\rho)^2}+\frac{B}{4}$

Table C2
The $I(2)$ and $I(4)$ integral expressions

	$I(2)$	$I(4)$
$\rho^2-1>0$	$\frac{1}{\rho^2}\left(-\frac{\sqrt{\rho^2-1}}{\rho}+\ln\left(1+\sqrt{\rho^2-1}\right)\right)$	$\frac{1}{\rho^2}\left(-\frac{\left(\sqrt{\rho^2-1}\right)^3}{\rho}+\frac{3}{2}\left(\rho\sqrt{\rho^2-1}-\ln\left(1+\sqrt{\rho^2-1}\right)\right)\right)$
$\rho^2-1<0$	$\frac{1}{\rho^2}\left(-\frac{\sqrt{1-\rho^2}}{\rho}+\cos^{-1}(\rho)\right)$	$\frac{1}{\rho^2}\left(-\frac{\left(\sqrt{1-\rho^2}\right)^3}{\rho}+\frac{3}{2}\left(-\rho\sqrt{1-\rho^2}+\cos^{-1}(\rho)\right)\right)$

$\psi^{\rho sph}(\beta,\alpha)\equiv\frac{1}{D^3(\beta,\alpha)}=R^{*^3}(\beta,\alpha)$, with $R^*(\beta,\alpha)$ the radius vector of the $V^*\frac{1}{\rho}$-spheroid. After the α integration over 2π around z which gives the axi-symmetric operator of Table 3, the $t^{\rho \mathrm{sph}}$ operator can be obtained from 2 integrals, for example $\cos^{2p}(\beta)=u^{2p}$, $p=1,2$

$$I(2)=2\int_0^1\frac{1}{2\rho^2}\left(\frac{1}{1+(\rho^2-1)u^2}\right)^{\frac{3}{2}}u^2\mathrm{d}u;\quad I(4)=2\int_0^1\frac{1}{2\rho^2}\left(\frac{1}{1+(\rho^2-1)u^2}\right)^{\frac{3}{2}}u^4\mathrm{d}u. \tag{C.2}$$

I(2) and I(4) depend on whether $\rho^2-1>0$ or $\rho^2-1<0$ (Table C2) and yield to the expressions of the Eshelby tensors for prolate and oblate spheroids (Christensen, 1979; Mura, 1987). For $\rho\to 1$, $I(2)\to 1/3$ and $I(4)\to 1/5$ respectively, giving $\mathbf{t}^{\mathrm{sph}}$, in Eqs. (18) and (19). When $\rho\to\infty$, $\mathbf{t}^{\rho \mathrm{sph}}\to\mathbf{t}^{\mathrm{cfib}}$ as defined of z axis, when $\rho\to 0$, $\mathbf{t}^{\rho \mathrm{sph}}\to\mathbf{t}^P(0,0)$, the platelet of z normal.

Appendix D. Non convex 2D and 3D bodies of constant total projection area

The Fig. D1 left shows (two examples of) a $\widehat{V}$ convex, semi regular even polygon built on a regular W odd one, in replacing linear edges by bi-linear elements, dLi say, and a $\breve{V}$ star shaped inclusion inscribed in W. $\breve{V}$ is built from replacing the W edges by the (−dLi) central symmetrals of the dLi elements, the dLi shape being limited by a no overlap request of the removed material from W. Due to identical additional projection contributions of a convex element with a planar boundary and of its central symmetral, compared to the projection of the planar boundary alone (Fig. D1 right), so related general $\widehat{V}$ and $\breve{V}$ shapes have equal $A_V(\omega)$ total projected area [11]. Similar constructions hold in R^3. Starting from a convex body of constant brightness, one can thus associate to it one or several non-convex bodies of same total projection area. Most celebrate convex bodies of constant brightness in R^2 are so-called N-orbiforms, one of which is drawn in Fig. D2 left, for $N=5$. In this figure are also drawn, non-convex shapes, obtained from

[11] The brightness of $\widehat{V}$ equals its total projection area, while the brightness of $\breve{V}$ is the one of its W convex hull.

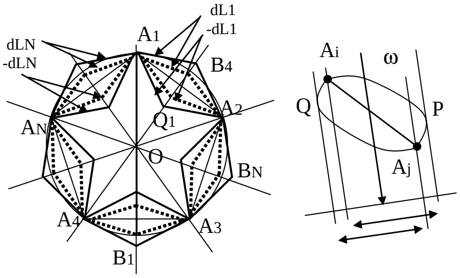

Fig. D1. Convex and non-convex polygons of same total projection area functions (full and dotted lines are two examples). Right, equal orthogonal ω-projections of two centrally reflected convex elements with one linear boundary part (proj($Q - Aj$) = proj($Ai - P$)).

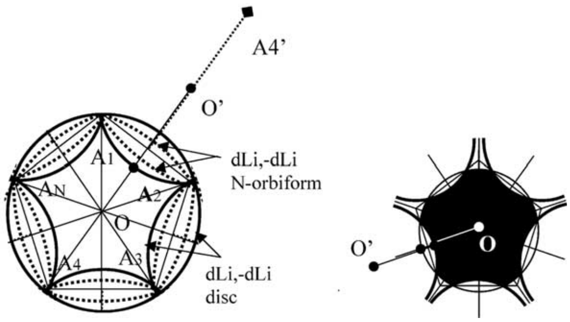

Fig. D2. Examples of 2D bodies with constant total projection area. Left, a N-orbiform and its opposite N-orbistar (dotted lines), a N-circular-star opposite to the disc (full lines). The two non-convex shapes are defined from inverted boundary elements of respectively the N-orbiform and the disc. Right, non-convex body with mixed convex and non-convex boundary elements of same radius.

reversing the boundary elements of a N-orbiform, and of a disc, with regard to the underlying N-polygon. Let call the former ones N-orbistars, and the latter ones (with $N \geqslant 5$ for no material overlap) N-circular stars. Infinitely many other non-convex bodies result from mixing convex and non-convex boundary parts (Fig. D2 right). All these shapes are "almost everywhere smooth", or are limits of everywhere smooth shapes, as when considering polyhedrons. In R^3, non-spherical convex bodies of constant brightness are constructed in Gardner (1995). Simple examples of non-convex ones of constant total projected area can be built from the sphere, creating "N-spherical stars" as the "N-circular stars" built in R^2 from a disc, in Fig. D2 left. This can be performed from using any inscribed convex polyhedron of the icosahedral/dodecahedral family, provided no material overlap, to invert, with respect to its faces, the spherical boundary elements delimited on the sphere by its vertices. As well, "gulf ball-like" R^3 bodies equivalent to the mixed shape of Fig. D2 right, of constant total projection area, can be built from the sphere. These R^2 and R^3 examples belong to the cfib and sphere $\widetilde{S}$—class respectively.

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